17^{th} **South Eastern European Mathematical Olympiad for University Students**
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SOLUTIONS OF THE PROBLEMS

Problem 1. Prove that if A and B are $n \times n$ square matrices with complex entries satisfying

$$A = AB - BA + A^2B - 2ABA + BA^2 + A^2BA - ABA^2$$

then $\det(A) = 0$.

Solution: 1. We have

$$A^k = A^kB - A^{k-1}BA + A^{k+1}B - A^kBA - A^kBA + A^{k-1}BA^2 + A^{k+1}BA - A^kBA^2.$$

Taking the trace and employing $\mathrm{tr}(MN) = \mathrm{tr}(NM)$ we deduce

$$\begin{aligned}\mathrm{tr}(A^k) = \mathrm{tr}(A^kB) - \mathrm{tr}((A^{k-1}B)A) + \mathrm{tr}(A^{k+1}B) - \mathrm{tr}((A^kB)A) - \mathrm{tr}((A^kB)A)\\ - \mathrm{tr}((A^{k-1}B)A^2) + \mathrm{tr}((A^{k+1}B)A) - \mathrm{tr}((A^kB)A^2) = 0.\end{aligned}$$

For any $k \geq 1$, $\mathrm{tr}(A^k) = 0$ and hence A is nilpotent. Therefore $\det(A) = 0$.

Solution: 2. If $\det(A) \neq 0$, multyplying the equation by A^{-1} from left (right), we get

$$I_n = B - A^{-1}BA + AB - 2BA + A^{-1}BA^2 + ABA - BA^2.$$

Taking trace and having in mind that $\mathrm{tr}(MN) = \mathrm{tr}(NM)$ we deduce:

$$\begin{aligned}n = \mathrm{tr}(I_n) = \mathrm{tr}(A(A^{-1}B)) - \mathrm{tr}((A^{-1}B)A) + \mathrm{tr}(AB) - \mathrm{tr}(BA) - \mathrm{tr}((BA^2)A^{-1}) +\\ + \mathrm{tr}(A^{-1}(BA^2)) + \mathrm{tr}(A(BA)) - \mathrm{tr}((BA)A) = 0,\end{aligned}$$

which is a contradiction. Hence $\det(A) = 0$.

Problem 2. For the sequence

$$S_n = \frac{1}{\sqrt{n^2+1^2}} + \frac{1}{\sqrt{n^2+2^2}} + \ldots + \frac{1}{\sqrt{n^2+n^2}},$$

find

$$\lim_{n\to\infty} n\left(n(\ln(1+\sqrt{2}) - S_n) - \frac{1}{2\sqrt{2}(1+\sqrt{2})}\right).$$

Solution: In what follows $O\left(x^k\right)$ stays for Cx^k where C is some constant.

$$f(x) = f(b) + f'(b)(x-b) + \frac{1}{2}f''(b)(x-b)^2 + \frac{1}{6}f'''(\theta)(x-b)^3$$

for some θ between a and b. It follows that

$$\int_a^b f(x)dx = f(b)(b-a) - \frac{1}{2}f'(b)(b-a)^2 + \frac{1}{6}f''(b)(b-a)^3 + O((b-a)^4). \qquad (1)$$

Now, let n be a positive integer. Then, for $k = 0, 1, 2, \ldots, n-1$,

$$\int_{(k-1)/n}^{k/n} f(x)dx = \frac{1}{n}f\left(\frac{k}{n}\right) - \frac{1}{2n^2}f'\left(\frac{k}{n}\right) + \frac{1}{6n^3}f''\left(\frac{k}{n}\right) + O\left(\frac{1}{n^4}\right). \qquad (2)$$

Summing over k then yields

$$\int_0^1 f(x)dx = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) - \frac{1}{2n^2}\sum_{k=1}^{n} f'\left(\frac{k}{n}\right) + \frac{1}{6n^3}\sum_{k=1}^{n} f''\left(\frac{k}{n}\right) + O\left(\frac{1}{n^3}\right). \qquad (3)$$

Similarly, we can get

$$f(1) - f(0) = \int_0^1 f'(x)dx = \frac{1}{n}\sum_{k=1}^{n} f'\left(\frac{k}{n}\right) - \frac{1}{2n^2}\sum_{k=1}^{n} f''\left(\frac{k}{n}\right) + O\left(\frac{1}{n^2}\right), \qquad (4)$$

and

$$f'(1) - f'(0) = \int_0^1 f''(x)dx = \frac{1}{n}\sum_{k=1}^{n} f''\left(\frac{k}{n}\right) + O\left(\frac{1}{n}\right). \qquad (5)$$

Combining (3), (4) and (5) we obtain

$$\int_0^1 f(x)dx = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) - \frac{1}{2n}(f(1)-f(0)) - \frac{1}{12n^2}(f'(1)-f'(0)) + O\left(\frac{1}{n^3}\right).$$

Now, let

$$f(x) = \frac{1}{\sqrt{1+x^2}}.$$

Then

$$\int_0^1 f(x)dx = \ln\left|x+\sqrt{1+x^2}\right|\Big|_0^1 = \ln(1+\sqrt{2}) - \ln(1) = \ln(1+\sqrt{2});$$

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \frac{1}{n}\sum_{k=1}^{n}\frac{1}{\sqrt{1+(k/n)^2}} = \sum_{k=1}^{n}\frac{1}{\sqrt{n^2+k^2}} = S_n;$$

$$f(1)-f(0) = \frac{1}{\sqrt{2}} - 1 = \frac{1-\sqrt{2}}{\sqrt{2}} = -\frac{1}{\sqrt{2}(1+\sqrt{2})};$$

$$f'(1)-f'(0) = -\frac{1}{2\sqrt{2}} - 0 = -\frac{1}{2\sqrt{2}}.$$

Hence

$$\ln(1+\sqrt{2}) = S_n + \frac{1}{2\sqrt{2}(1+\sqrt{2})n} + \frac{1}{24\sqrt{2}n^2} + O\left(\frac{1}{n^3}\right).$$

Finally,

$$\lim_{n\to\infty} n\left(n(\ln(1+\sqrt{2})-S_n) - \frac{1}{2\sqrt{2}(1+\sqrt{2})}\right) = \frac{1}{24\sqrt{2}}.$$

Problem 3. Prove that: if A is $n \times n$ square matrix with complex entries such that $A + A^* = A^2\, A^*$, then $A = A^*$. (For any matrix M, denote by $M^* = \overline{M}^t$ the conjugate transpose of M.)

Solution: We show first that A is normal, i.e., $A\, A^* = A^*\, A$.

We have that $A + A^* = A^2\, A^*$ leads to $A = (A^2 - I_n)A^* \quad (1)$, hence $A \pm I_n = (A - I_n)(A + I_n)A^* \pm I_n$, so

$$(A - I_n)\,[(A + I_n)A^* - I_n] = I_n$$
$$(A + I_n)\,[I_n - (A - I_n)A^*] = I_n,$$

which leads to $A - I_n$ and $A + I_n$ being invertible. From here, $A^2 - I_n$ is also invertible, and by (1) it follows that $A^* = (A^2 - I_n)^{-1}A$. Using the Cayley–Hamilton theorem, it follows that $(A^2 - I_n)^{-1}$ is a polynomial of $A^2 - I_n$, hence a polynomial of A, so $A^*\, A = A\, A^*$.

Since A is normal, it is unitary diagonalizable, i.e., there exist a unitary matrix $U \in \mathcal{M}_n(\mathbb{C})$ and $D = \operatorname{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n]$ a diagonal matrix such that $A = UDU^*$. Then $A^* = U\overline{D}U^*$, which, by the hypothesis leads to $D + \overline{D} = D^2\overline{D}$, meaning that $\lambda_i + \overline{\lambda_i} = \lambda_i^2\overline{\lambda_i}$, for all $i \in \{1, 2, \ldots, n\}$. Then $2\operatorname{Re}\lambda_i = \lambda_i \cdot |\lambda_i|^2$, so λ_i are all real, and $D = \overline{D}$. This is now enough for $A = A^*$.

Problem 4. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous, strictly decreasing function such that $f([0,1])\subseteq[0,1]$.

(i) For all $n\in\mathbb{N}\backslash\{0\}$, prove that there exists a unique $a_n\in(0,1)$, solution of the equation

$$f(x)=x^n.$$

Moreover, if (a_n) is the sequence defined as above, prove that $\lim\limits_{n\to\infty}a_n=1$.

(ii) Suppose f has a continuous derivative, with $f(1)=0$ and $f'(1)<0$. For any $x\in\mathbb{R}$, we define

$$F(x)=\int_x^1 f(t)\ dt.$$

Study the convergence of the series $\sum_{n=1}^{\infty}(F(a_n))^{\alpha}$, with $\alpha\in\mathbb{R}$.

Solution: (i) Consider the continuous function $g:[0,1]\to\mathbb{R}$ given by $g(x)=f(x)-x^n$, and observe that $g(0)=f(0)>0$, and $g(1)=f(1)-1<0$. It follows the existence of $a_n\in(0,1)$ such that $g(a_n)=0$. For uniqueness, observe that if would exists two solutions of the equation (4), say $a_n<b_n$, we would obtain

$$f(a_n)>f(b_n)\Leftrightarrow a_n^n>b_n^n\Leftrightarrow a_n>b_n,$$

a contradiction.

We prove that the sequence (a_n) is strictly increasing. If it would exist $n\in\mathbb{N}^*$ such that $a_n\geq a_{n+1}$, we would obtain that

$$f(a_n)\leq f(a_{n+1})\Leftrightarrow a_n^n\leq a_{n+1}^{n+1}<a_{n+1}^n,$$

since f is strictly decreasing and $a_{n+1}\in(0,1)$. It follows that $a_n<a_{n+1}$, a contradiction. Hence, (a_n) is strictly increasing and bounded above by 1, so it converges to $\ell\in(0,1]$. Suppose, by contradiction, that $\ell<1$. Since $f(a_n)=a_n^n$ for any n, using the continuity of f it follows that $f(\ell)=0$ for $\ell<1$, contradicting the fact that f is strictly decreasing with $f(1)\geq 0$. Hence, $\lim\limits_{n\to\infty}a_n=1$.

(ii) Observe that F is well-defined, of class C^2, with $F(1)=0$, $F'(x)=-f(x)\Rightarrow F'(1)=0$, $F''(x)=-f'(x)\Rightarrow F''(1)>0$. Moreover, remark that $F(x)>0$ on $[0,1)$. Using the Taylor formula on the interval $[a_n,1]$, it follows that for any n, there exist $c_n,d_n\in(a_n,1)$ such that

$$\begin{aligned} F(a_n) &= F(1)+F'(1)(a_n-1)+\frac{F''(c_n)}{2}(a_n-1)^2=\frac{F''(c_n)}{2}(a_n-1)^2,\\ f(a_n) &= f(1)+f'(d_n)(a_n-1)=f'(d_n)(a_n-1). \end{aligned} \tag{1}$$

Hence, since $c_n\to 1$ and F is C^2, we obtain

$$\lim_{n\to\infty}\frac{(1-a_n)^2}{F(a_n)}=\frac{2}{F''(1)}\in(0,+\infty),$$

so due to the comparison test,

$$\sum_{n=1}^{\infty}\left(F\left(a_n\right)\right)^{\alpha}\sim\sum_{n=1}^{\infty}\left(1-a_n\right)^{2\alpha}.$$

But

$$\begin{aligned}\lim_{n\to\infty}n\left(1-a_n\right)&=-\lim_{n\to\infty}n\cdot\frac{\left(a_n-1\right)}{\ln\left(1+\left(a_n-1\right)\right)}\cdot\ln a_n\\&=-\lim_{n\to\infty}\ln a_n^n=-\lim_{n\to\infty}\ln f\left(a_n\right)=-\ln\left(\lim_{n\to\infty}f\left(a_n\right)\right)=+\infty.\end{aligned}$$

It follows that $\sum_{n=1}^{\infty}\left(1-a_n\right)$ diverges and, furthermore, $\sum_{n=1}^{\infty}\left(1-a_n\right)^{2\alpha}$ diverges for any $2\alpha\le 1$.

Next, consider arbitrary $\gamma\in(0,1)$. Using (1) and the fact that $d_n\to 1$, we obtain

$$\begin{aligned}&\lim_{n\to\infty}n^{\gamma}\left(1-a_n\right)=\lim_{n\to\infty}\left[n\left(1-a_n\right)\right]^{\gamma}\cdot\left(1-a_n\right)^{1-\gamma}\\&=\lim_{n\to\infty}\left[n\left(1-a_n\right)\right]^{\gamma}\cdot\left[\frac{f\left(a_n\right)}{-f'\left(d_n\right)}\right]^{1-\gamma}=\frac{1}{\left(-f'\left(1\right)\right)^{1-\gamma}}\cdot\lim_{n\to\infty}\left[-\ln f\left(a_n\right)\right]^{\gamma}\cdot\left[e^{\ln f\left(a_n\right)}\right]^{1-\gamma}.\end{aligned}$$

Observe that

$$-\ln f\left(a_n\right)\to+\infty\quad\text{and}\quad\lim_{x\to+\infty}\frac{x^{\gamma}}{e^{(1-\gamma)x}}=0,$$

hence $\lim\limits_{n\to\infty}n^{\gamma}\left(1-a_n\right)=0$. So, if $\alpha>\dfrac{1}{2}$, we obtain that there exists $\varepsilon>0$ such that $2\alpha>1+\varepsilon$, hence for $\gamma:=\dfrac{1+\varepsilon}{2\alpha}<1$, we get

$$\lim_{n\to\infty}n^{2\alpha\gamma}\left(1-a_n\right)^{2\alpha}=\lim_{n\to\infty}n^{(1+\varepsilon)}\left(1-a_n\right)^{2\alpha}=0.$$

Using the comparison test, it follows that the series $\sum_{n=1}^{\infty}\left(1-a_n\right)^{2\alpha}$ converges. In conclusion, the series $\sum_{n=1}^{\infty}\left(F\left(a_n\right)\right)^{\alpha}$ converges iff $\alpha>\frac{1}{2}$.