PROBLEMS AND SOLUTIONS FOR 25th BALKAN MATHEMATICAL OLYMPIAD

Problem 1

An acute-angled scalene triangle ABC is given, with $AC > BC$. Let O be its circumcentre, H its orthocentre, and F the foot of the altitude from C. Let P be the point (other than A) on the line AB such that $AF=PF$, and M be the midpoint of AC. We denote the intersection of PH and BC by X, the intersection of OM and FX by Y, and the intersection of OF and AC by Z. Prove that the points F, M, Y and Z are concyclic.

Solution:

It is enough to show that $OF \perp FX$.

Let $OE \perp AB$, then it is trivial that :

$$CH = 2OE. \qquad (1)$$

Since from the hypothesis we have $PF = AF$ then we take $PB = PF - BF$ or

$$PB = AF - BF \qquad (2)$$

Also, $\angle XPB = \angle HAP$ and $\angle HAP = \angle HCX$ since AFGC in inscribable (where G is the foot of the altidude from A),

so $\angle XPB = \angle HCX$ and since $\angle BXP = \angle HXC$, the triangles XHC and XBP are similar.

If XL and XD are respectively the heights of the triangles XHC and XBP we have:

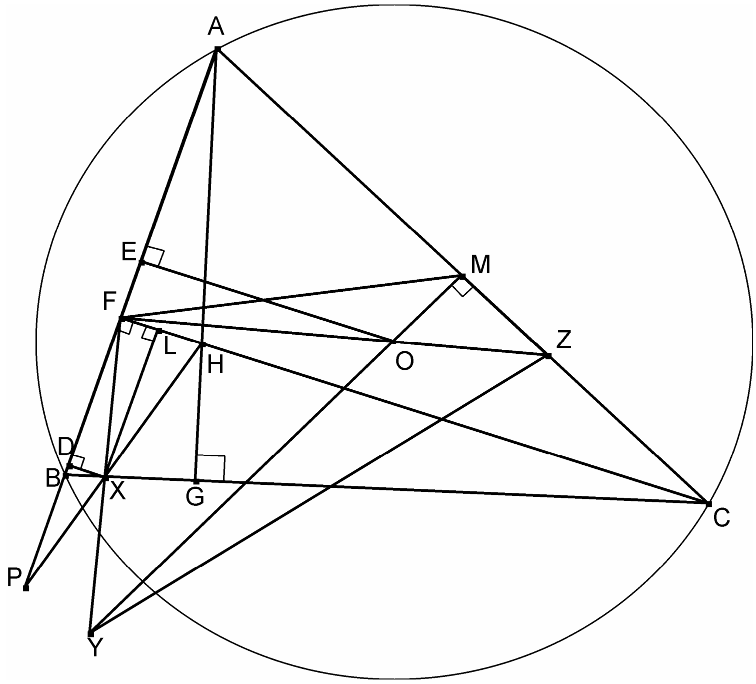
$$\frac{XD}{XL} = \frac{PB}{CH},$$

and from (1) and (2) we get:

$$\frac{XD}{XL} = \frac{AF - BF}{2OE} = \frac{FE}{OE} \Rightarrow \frac{XD}{FD} = \frac{FE}{OE}$$

Therefore the triangles XFD, OEF are similar and we get:

$\angle OFX = \angle OFC + \angle LFX = \angle FOE + \angle FXD = \angle XFD + \angle FXD = 90^o$, so $OF \perp FX$.



Problem 2
Does there exist a sequence $a_1, a_2, ..., a_n, ...$ of positive real numbers satisfying both of the following conditions:

(i) $\sum_{i=1}^{n} a_i \le n^2$, for every positive integer n;

(ii) $\sum_{i=1}^{n} \frac{1}{a_i} \le 2008$, for every positive integer n?

Solution.
The answer is no.
It is enough to show that

if $\sum_{i=1}^{n} a_i \le n^2$ for any n, then $\sum_{i=2}^{2^n} \frac{1}{a_i} > \frac{n}{4}$. (or any other precise estimate)

For this, we use that $\sum_{i=2^k+1}^{2^{k+1}} a_i \sum_{i=2^k+1}^{2^{k+1}} \frac{1}{a_i} \ge 2^{2k}$ for any $k \ge 0$ by the arithmetic-harmonic mean inequality.

Since $\sum_{i=2^k+1}^{2^{k+1}} a_i < \sum_{i=1}^{2^{k+1}} a_i \le 2^{2k+2}$, it follows that $\sum_{i=2^k+1}^{2^{k+1}} \frac{1}{a_i} > \frac{1}{4}$

and hence $\sum_{i=2}^{2^n} \frac{1}{a^i} > \sum_{k=0}^{n-1} \sum_{i=2^k+1}^{2^{k+1}} \frac{1}{a_i} > \frac{n}{4}$. (it can be stated in words)

Remark: no points for using some inequality, that doesn't lead to solution

Problem 3

Let n be a positive integer. The rectangle $ABCD$ with side lengths $AB = 90n+1$ and $BC = 90n+5$ is partitioned into unit squares with sides parallel to the sides of $ABCD$. Let S be the set of all points which are vertices of these unit squares. Prove that the number of lines which pass through at least two points from S is divisible by 4.

Solution.

Denote $90n+1 = m$. We investigate the number of the lines modulo 4 consecutively reducing different types of lines.

The vertical and horizontal lines are

$(m+5)+(m+1) = 2(m+3)$ which is divisible to 4.

Moreover, every line which makes an acute angle to the axe Ox (i.e. that line has a positive angular coefficient) corresponds to unique line with an obtuse angle (consider the symmetry with respect to the line through the midpoints of AB and CD). Therefore it is enough to prove that the lines with acute angles are an even number.

Every line which does not pass through the center O of the rectangle corresponds to another line with the same angular coefficent(consider the symmetry with respect to O). Therefore it is enough to consider the lines through O.

Every line through O has an angular coefficient $\frac{p}{q}$, where $(p,q)=1$, p and q are odd positive integers. (To see this, consider the two nearest, from the two sides, to O points of the line).

If $p \neq 1$, $q \neq 1$, $p \leq m$ and $q \leq m$, the line with angular coefficient $\frac{p}{q}$, uniquely corresponds to the line with angular coefficient $\frac{q}{p}$. It remains to prove that the number of the remaining lines is even.

The last number is

$$1+\frac{\varphi(m+2)}{2}+\frac{\varphi(m+4)}{2}-1=\frac{\varphi(m+2)+\varphi(m+4)}{2}$$

because we have:

1) one line with $p=q=1$;
2) $\frac{\varphi(m+2)}{2}$ lines with angular coefficient $\frac{p}{m+2}$, $p \leq m$ is odd and $(p, m+2)=1$;
3) $\frac{\varphi(m+4)}{2}-1$ lines with angular coefficient $\frac{p}{m+4}$, $p \leq m$ is odd and $(p, m+4)=1$.

Now the assertion follows from the fact that the number $\varphi(m+2)+\varphi(m+4)=\varphi(90n+3)+\varphi(90n+5)$ is divisible to 4.

Problem 4

Let c be a positive integer. The sequence $a_1, a_2, ..., a_n, ...$ is defined by $a_1 = c$, and $a_{n+1} = a_n^2 + a_n + c^3$, for every positive integer n. Find all values of c for which there exist some integers $k \geq 1$ and $m \geq 2$, such that $a_k^2 + c^3$ is the m^{th} power of some positive integer.

Solution.

First, notice:

$$a_{n+1}^2 + c^3 = (a_n^2 + a_n + c^3)^2 + c^3 = (a_n^2 + c^3)(a_n^2 + 2a_n + 1 + c^3)$$

We first prove that $a_n^2 + c^3$ and $a_n^2 + 2a_n + 1 + c^3$ are coprime.

We prove by induction that $4c^3 + 1$ is coprime with $2a_n + 1$, for every $n \geq 1$.

Let $n = 1$ and p be a prime divisor of $4c^3 + 1$ and $2a_1 + 1 = 2c + 1$. Then p divides $2(4c^3 + 1) = (2c + 1)(4c^2 - 2c + 1) + 1$, hence p divides 1, a contradiction. Assume now that $(4c^3 + 1, 2a_n + 1) = 1$ for some $n \geq 1$ and the prime p divides $4c^3 + 1$ and $2a_{n+1} + 1$. Then p divides $4a_{n+1} + 2 = (2a_n + 1)^2 + 4c^3 + 1$, which gives a contradiction.

Assume that for some $n \geq 1$ the number

$$a_{n+1}^2 + c^3 = (a_n^2 + a_n + c^3)^2 + c^3 = (a_n^2 + c^3)(a_n^2 + 2a_n + 1 + c^3)$$

is a power. Since $a_n^2 + c^3$ and $a_n^2 + 2a_n + 1 + c^3$ are coprime,

than $a_n^2 + c^3$ is a power as well.

The same argument can be further applied giving that $a_1^2 + c^3 = c^2 + c^3 = c^2(c + 1)$ is a power.

If $a^2(a + 1) = t^m$ with odd $m \geq 3$, then $a = t_1^m$ and $a + 1 = t_2^m$, which is impossible. If $a^2(a + 1) = t^{2m_1}$ with $m_1 \geq 2$, then $a = t_1^{m_1}$ and $a + 1 = t_2^{m_1}$, which is impossible.

Therefore $a^2(a + 1) = t^2$ whence we obtain the solutions $a = s^2 - 1$, $s \geq 2$, $s \in \mathbb{N}$.