Bucharest, March 4th, 2011

SOUTH EASTERN EUROPEAN MATHEMATICAL OLYMPIAD FOR UNIVERSITY STUDENTS

PROBLEMS

Problem 1 *For a given integer $n \geq 1$, let $f : [0,1] \to \mathbb{R}$ be a non-decreasing function. Prove that*

$$\int_0^1 f(x)\,\mathrm{d}x \leq (n+1)\int_0^1 x^n f(x)\,\mathrm{d}x.$$

Find all non-decreasing continuous functions for which equality holds.

Problem 2 *Let $A = (a_{ij})$ be a real $n \times n$ matrix such that $A^n \neq 0$ and $a_{ij}a_{ji} \leq 0$ for all i, j. Prove that there exist two nonreal numbers among eigenvalues of A.*

Problem 3 *Given vectors $\overline{a}, \overline{b}, \overline{c} \in \mathbb{R}^n$, show that*

$$\left(||\overline{a}||\left\langle \overline{b}, \overline{c}\right\rangle\right)^2 + \left(||\overline{b}||\left\langle \overline{a}, \overline{c}\right\rangle\right)^2 \leq ||\overline{a}||||\overline{b}||\left(||\overline{a}||||\overline{b}|| + |\left\langle \overline{a}, \overline{b}\right\rangle|\right)||\overline{c}||^2,$$

where $\langle \overline{x}, \overline{y}\rangle$ denotes the scalar (inner) product of the vectors $\overline{x}$ and $\overline{y}$ and $||\overline{x}||^2 = \langle \overline{x}, \overline{x}\rangle$.

Problem 4 *Let $f : [0,1] \to \mathbb{R}$ be a twice continuously differentiable increasing function. Define the sequences given by $L_n = \frac{1}{n}\sum\limits_{k=0}^{n-1} f\left(\frac{k}{n}\right)$ and $U_n = \frac{1}{n}\sum\limits_{k=1}^{n} f\left(\frac{k}{n}\right)$, $n \geq 1$. The interval $[L_n, U_n]$ is divided into three equal segments. Prove that, for large enough n, the number $I = \int\limits_0^1 f(x)\,\mathrm{d}x$ belongs to the middle one of these three segments.*

Each problem is 10 points worth.
Allowed time: 5 hours.