Problem 1.

Solution. As $a^3b - 1 = b(a^3+1) - (b+1)$ and $a+1 \mid a^3+1$, we have $a+1 \mid b+1$.

As $b^3a + 1 = a(b^3-1) + (a+1)$ and $b-1 \mid b^3-1$, we have $b-1 \mid a+1$.

So $b-1 \mid b+1$ and hence $b-1 \mid 2$.

- If $b = 2$, then $a+1 \mid b+1 = 3$ gives $a = 2$. Hence $(a,b) = (2,2)$ is the only solution in this case.
- If $b = 3$, then $a+1 \mid b+1 = 4$ gives $a = 1$ or $a = 3$. Hence $(a,b) = (1,3)$ and $(3,3)$ are the only solutions in this case.

To summarize, $(a,b) = (1,3)$, $(2,2)$ and $(3,3)$ are the only solutions.

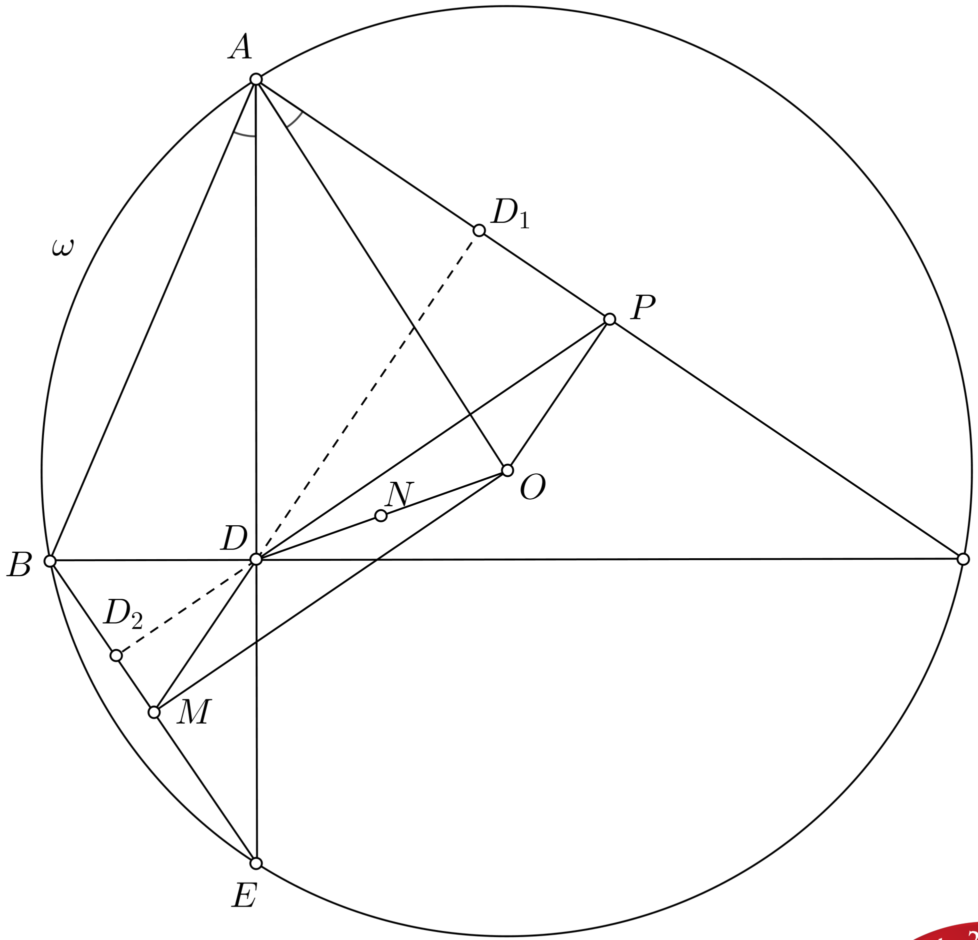
Problem 2.

Solution. We will show that $MOPD$ is a parallelogram. From this it follows that M, N, P are collinear.

Since $\angle BAD = \angle CAO = 90° - \angle ABC$, D is the foot of the perpendicular from A to side BC. Since M is the midpoint of the line segment BE, we have $BM = ME = MD$ and hence $\angle MDE = \angle MED = \angle ACB$.

Let the line MD intersect the line AC at D_1. Since $\angle ADD_1 = \angle MDE = \angle ACD$, MD is perpendicular to AC. On the other hand, since O is the center of the circumcircle of triangle ABC and P is the midpoint of the side AC, OP is perpendicular to AC. Therefore MD and OP are parallel.

Similarly, since P is the midpoint of the side AC, we have $AP = PC = DP$ and hence $\angle PDC = \angle ACB$. Let the line PD intersect the line BE at D_2. Since $\angle BDD_2 = \angle PDC = \angle ACB = \angle BED$, we conclude that PD is perpendicular to BE. Since M is the midpoint of the line segment BE, OM is perpendicular to BE and hence OM and PD are parallel.



Problem 3.

Solution 1. By the AM-GM Inequality we have:

$$\frac{a+1}{2} + \frac{2}{a+1} \geq 2$$

Therefore

$$a + 2b + \frac{2}{a+1} \geq \frac{a+3}{2} + 2b\,.$$

and, similarly,

$$b + 2a + \frac{2}{b+1} \geq 2a + \frac{b+3}{2}\,.$$

On the other hand,

$$(a + 4b + 3)(b + 4a + 3) \geq (\sqrt{ab} + 4\sqrt{ab} + 3)^2 \geq 64$$

by the Cauchy-Schwarz Inequality as $ab \geq 1$, and we are done.

Solution 2. Since $ab \geq 1$, we have $a + b \geq a + 1/a \geq 2\sqrt{a \cdot (1/a)} = 2$.

Then

$$\begin{aligned} a + 2b + \frac{2}{a+1} &= b + (a+b) + \frac{2}{a+1} \\ &\geq b + 2 + \frac{2}{a+1} \\ &= \frac{b+1}{2} + \frac{b+1}{2} + 1 + \frac{2}{a+1} \\ &\geq 4\sqrt[4]{\frac{(b+1)^2}{2(a+1)}} \end{aligned}$$

by the AM-GM Inequality. Similarly,

$$b + 2a + \frac{2}{b+1} \geq 4\sqrt[4]{\frac{(a+1)^2}{2(b+1)}}\,.$$

Now using these and applying the AM-GM Inequality another time we obtain:

$$\begin{aligned}\left(a+2b+\frac{2}{a+1}\right)\left(b+2a+\frac{2}{b+1}\right) &\geq 16\sqrt[4]{\frac{(a+1)(b+1)}{4}}\\ &\geq 16\sqrt[4]{\frac{(2\sqrt{a})(2\sqrt{b})}{4}}\\ &= 16\sqrt[8]{ab}\\ &\geq 16\end{aligned}$$

Solution 3. We have

$$\begin{aligned}\left(a+2b+\frac{2}{a+1}\right)\left(b+2a+\frac{2}{b+1}\right) &= \left((a+b)+b+\frac{2}{a+1}\right)\left((a+b)+a+\frac{2}{b+1}\right)\\ &\geq \left(a+b+\sqrt{ab}+\frac{2}{\sqrt{(a+1)(b+1)}}\right)^2\end{aligned}$$

by the Cauchy-Schwarz Inequality.

On the other hand,

$$\frac{2}{\sqrt{(a+1)(b+1)}} \geq \frac{4}{a+b+2}$$

by the AM-GM Inequality and

$$a+b+\sqrt{ab}+\frac{2}{\sqrt{(a+1)(b+1)}} \geq a+b+1+\frac{4}{a+b+2} = \frac{(a+b+1)(a+b-2)}{a+b+2}+4 \geq 4$$

as $a+b \geq 2\sqrt{ab} \geq 2$, finishing the proof.

Problem 4.

Solution. a. Yes. Let $a \leq b \leq c \leq d \leq e$ be the numbers chosen by Alice. As each number appears in a pairwise sum 4 times, by adding all 10 pairwise sums and dividing the result by 4, Bob obtains $a+b+c+d+e$. Subtracting the smallest and the largest pairwise sums $a+b$ and $d+e$ from this he obtains c. Subtracting c from the second largest pairwise sum $c+e$ he obtains e. Subtracting e from the largest pairwise sum $d+e$ he obtains d. He can similarly determine a and b.

b. Yes. Let $a \leq b \leq c \leq d \leq e \leq f$ be the numbers chosen by Alice. As each number appears in a pairwise sum 5 times, by adding all 15 pairwise sums and dividing the result by 5, Bob obtains $a+b+c+d+e+f$. Subtracting the smallest and the largest pairwise sums $a+b$ and $e+f$ from this he obtains $c+d$.

Subtracting the smallest and the second largest pairwise sums $a+b$ and $d+f$ from $a+b+c+d+e+f$ he obtains $c+e$. Similarly he can obtain $b+d$. He uses these to obtain $a+f$ and $b+e$.

Now $a+d$, $a+e$, $b+c$ are the three smallest among the remaining six pairwise sums. If Bob adds these up, subtracts the known sums $c+d$ and $b+e$ from the result and divides the difference by 2, he obtains a. Then he can determine the remaining numbers.

c. No. If Alice chooses the eight numbers $1, 5, 7, 9, 12, 14, 16, 20$, then Bob cannot be sure to guess these numbers correctly as the eight numbers $2, 4, 6, 10, 11, 15, 17, 19$ also give exactly the same 28 pairwise sums as these numbers.