

Language: English

Saturday, April 28, 2012

Problem 1. Let A, B and C be points lying on a circle Γ with centre O. Assume that $\angle ABC > 90^{\circ}$. Let D be the point of intersection of the line AB with the line perpendicular to AC at C. Let ℓ be the line through D which is perpendicular to AO. Let E be the point of intersection of ℓ with the line AC, and let E be the point of intersection of E with E that lies between E and E.

Prove that the circumcircles of triangles BFE and CFD are tangent at F.

Problem 2. Prove that

$$\sum_{\text{cyc}} (x+y)\sqrt{(z+x)(z+y)} \ge 4(xy+yz+zx),$$

for all positive real numbers x, y, and z.

The notation above means that the left-hand side is

$$(x+y)\sqrt{(z+x)(z+y)} + (y+z)\sqrt{(x+y)(x+z)} + (z+x)\sqrt{(y+z)(y+x)}$$
.

Problem 3. Let n be a positive integer. Let $P_n = \{2^n, 2^{n-1} \cdot 3, 2^{n-2} \cdot 3^2, \dots, 3^n\}$. For each subset X of P_n , we write S_X for the sum of all elements of X, with the convention that $S_\emptyset = 0$ where \emptyset is the empty set. Suppose that y is a real number with $0 \le y \le 3^{n+1} - 2^{n+1}$.

Prove that there is a subset Y of P_n such that $0 \le y - S_Y < 2^n$.

Problem 4. Let \mathbb{Z}^+ be the set of positive integers. Find all functions $f: \mathbb{Z}^+ \to \mathbb{Z}^+$ such that the following conditions both hold:

- (i) f(n!) = f(n)! for every positive integer n,
- (ii) m-n divides f(m)-f(n) whenever m and n are different positive integers.

Each problem is worth 10 points. Time allowed: 4 hours and 30 minutes.